BRICK WALL AND QUANTUM STATISTICAL ENTROPY OF BLACK HOLE

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We discuss the statistical-mechanical entropy of black hole calculated according to 't Hooft. It is argued that in presence of horizon the statistical mechanics of quantum fields depends on their UV behavior. The "brick wall" model was shown to provide a correct description when the "brick wall" parameter is less than any UV cut-off.

Since Bekenstein introduced the thermodynamical analogy in black hole physics and Hawking discovered thermal radiation from a black hole confirming this analogy [1], it is an intriguing problem as to what degrees of freedom are counted by the entropy of a black hole. Equivalently, what (if any) statistical mechanics is responsible for the Bekenstein-Hawking entropy?

According to 't Hooft [2] the statistical-mechanical entropy S_{SM} arises from a thermal bath of quantum fields propagating outside the horizon. It should be noted that every calculation of statistical entropy encounters the problem of dealing with the very peculiar behavior of the physical quantities near the horizon where they typically diverge. To remove these divergences 't Hooft introduced "brick wall": a fixed boundary near the horizon within which the quantum field does not propagate. Essentially, this procedure (as it formulated in [2]) must be implemented in addition to the removing of standard ultra-violet divergences. Contrary to statistical entropy the thermodynamical entropy of black hole is finite (after UV-renormalization) quantity not possessing any kind of the "brick wall" divergence. This fact has inspired to argue in [3] that black holes provide us with a unique exapmle of a specific system for which these entropies do not necessarily coincide.

However, dealing with *quantum* field, that is system having infinite number of degrees of freedom, we have to take into account that not all of these freedoms are, in fact, physical. Indeed, the high energy modes lead to unphysical infinities and must be subtracted. The well-defined UV-renormalization is invariant procedure of such a subtraction. Therefore, formulating the statistical mechanics for a quantim field we need in the similar subtraction of unphysical modes which must be excluded in the statistical ensemble. How this really happens was demonstrated in [4] by using the Pauli-Villars (PV) regularization scheme. It consists in introducing a number of fictitious fields (regulators) of

different statistics and with very large masses. Remarkably, this procedure not only yields the standard UV regularization but automatically implements a cut-off for the entropy calculation allowing to remove “brick wall”.

The interplay of “brick wall” and UV regularization is easily seen for two-dimensional massless scalar field. Applying Pauli-Villars regularization in two dimensions one needs to introduce a set of fictitious fields with very large masses: two anticommuting scalar fields with mass $\mu_{1,2} = \mu$ and one commuting field with mass $\mu_3 = \sqrt{2}\mu$. Consider the free energy of the ensemble of the original scalar field and regulators with an inverse temperature β:

$$\beta F = \sum_n \ln(1 - e^{-\beta E_n}) \tag{1}$$

Note that energy E_n in (1) is defined with respect to Killing vector ∂_t $(\tau = \imath t)$ and fields are expanded as $\phi = e^{\imath Et} f(x)$. Therefore, β in (1) is related with temperature T measured at $x = L$ as $T^{-1} = \beta g^{1/2}(L)$. The relevant density matrix is $\rho = \sum_n \phi_n \phi_n^* e^{-\beta E_n}$, where $\{\phi_n\}$ is basis of eigen-vectors. One should take into account that for the regulator fields the Hilbert space hase indefinite metric and hence a part of regulators contributes with minus sign.

The free energy (1) can be determined for the arbitrary black hole metric $ds^2 = -g dt^2 + g^{-1} dx^2$ without reference to the precise form of the metric function $g(x)$. Repeating the calculation of ref.[4] in this 2D case and applying WKB approximation we finally get

$$F = -\frac{1}{\pi} \int_0^\infty \frac{dE}{e^{\beta E} - 1} \int_{x_+ + h}^{L} \frac{dx}{g(x)} (E - 2(E^2 - \mu^2 g(x))^{1/2}$$
$$+ (E^2 - 2\mu^2 g(x))^{1/2}) \quad . \tag{2}$$

It should be noted that the WKB approximation for the original massless scalar field is really exact. We introduced in (2) a “brick wall” cut-off h. In fact, one can see that divergences at small h are precisely cancelled in (2) between the original scalar and the regulator fields. This is 2D analog of the mechanism discovered in [4]. So one can remove the cut-off in (2). However we will keep it arbitrarily small in the process of calculation of separate terms entering in (2).

It is straightforward to compute the contribution of the original massless field in (2). For computation of the regulator's contribution take the fixed E and consider the integral:

$$I[\mu] = \int_{x_+ + h}^{L_E} \frac{dx}{g(x)} (E^2 - \mu^2 g(x))^{1/2} \quad , \tag{3}$$

where integration is doing from the horizon $(x_+ + h)$ to distance L_E defined from equation $g(L_E) = \frac{E^2}{\mu^2}$. It is clear that when μ grows L_E becomes closer and closer to $(x_+ + h)$. So, considering limit of large μ we conclude that integral (3) is concentrated near the horizon where we have: $g(x) = \frac{4\pi}{\beta_H}(x - x_+) = (\frac{2\pi\rho}{\beta_H})^2$, $\frac{dx}{g} = \frac{\beta_H}{2\pi}\frac{d\rho}{\rho}$ and the new radial variable ρ now runs from $\epsilon = \sqrt{\frac{\beta_H h}{\pi}}$ to $(\frac{E\beta_H}{2\pi\mu})$. The integral (3) then in the limit of small ϵ reads:

$$
\begin{aligned}
I[\mu] &= \mu \int_{\epsilon}^{\frac{E\beta_H}{2\pi\mu}} \frac{d\rho}{\rho}\sqrt{(\frac{E\beta_H}{2\pi\mu})^2 - \rho^2} \\
&= -\frac{(E\beta_H)}{2\pi}(1 + \frac{1}{2}\ln 2 + \ln(\frac{\mu\epsilon\pi}{E\beta_H})) \quad . \qquad (4)
\end{aligned}
$$

This is the key identity allowing computation of the free energy (2). Omitting details which are rather simple the result is

$$
F = -\frac{1}{12}[\frac{\beta_H}{2\beta^2}\int_{x_+}^{L}\frac{dx}{g}(\frac{4\pi}{\beta_H} - g') + \frac{\beta_H}{\beta^2}\ln(\mu\beta g^{1/2}(L))] + \frac{\beta_H}{\beta^2}C \quad , \qquad (5)
$$

where we removed the brick wall cut-off and used that $\int_0^\infty \frac{dx x}{e^x - 1} = \frac{\pi^2}{6}$. The statistical-mechanical free energy (5) is really an off-shell quantity (see [4]) defined for arbitrary black hole metric and β not necessarily equal to β_H.

Calculating now entropy $S_{ST} = \beta^2\partial_\beta F$ and putting $\beta = \beta_H$ we obtain:

$$
S_{ST} = \frac{1}{12}\int_{x_+}^{L}\frac{dx}{g}(\frac{4\pi}{\beta_H} - g') + \frac{1}{6}\ln(\mu\beta_H g^{1/2}(L)) + C \quad , \qquad (6)
$$

where C is some numerical constant not depending on μ or metric $g(x)$. As was demonstrated in[6] $S_{ST}(6)$ exactly coincides with thermodynamical entropy of quantum field. So, at least this part of the thermodynamical entropy has the statistical meaning.

It is important to note the crucial interplay of two different limits $h \to 0$ (brick wall) and $\mu^{-1} \to 0$ (UV-regulatorization). If one takes the limit $\mu^{-1} \to$ first one obtains that contribution of the regulators in the free energy (2) completely vanishes. On then gets the quantities which are functions of the brick wall parameter h and divergent in the limit $h \to 0$. These are that quantities calculated in [3]. Elimination of their divergence (with respect to limit $h \to 0$) might require some subtraction procedure proposed in [3]. Note, that in this regime the "brick wall" is treated as real boundary staying at

macroscopical distance h from the horizon with h being larger than any UV cut-off μ^{-1}. However, in this case, this is no more a black hole.

The situation is different if we consider "brick wall" as an fictitious imaginary boundary with h being smaller any scale μ^{-1} of UV cut-off. Then the "brick wall" divergences are eliminated by the standard UV-regularization and the UV-regulators do contribute to the free energy and entropy. This contribution is concentrated at the horizon. It leads to appearance of additional terms ($\int g^{-1}g'$) in the entropy (6) that are finite after renormalization. It is worth noting that mechanism of this phenomenon is similar to that of the conformal anomaly. This similarity is not occasional since the result for the statistical entropy (6) occurs to coincide with the thermodynamical expression which is indeed originated from the conformal anomaly of the Polyakov-Liouville action. We do not have this phenomenon in the statistical mechanics on space-time without horizons where the statistical entropy was proved to be conformal invariant and not dependent on UV cut-off (see [5]). This is easily seen from our analysis. Indeed, in this case we have $g(x) \geq g_0 > 0$ everywhere and for large UV cut-off $\mu > \mu_0 = \frac{E}{g_0^{1/2}}$ contribution of the regulators disappears in the free energy (2). Thus, in the presence of horizons the statistical mechanics of quantum fields depends on their UV behavior. The UV-regulators lead to non-trivial contribution to statistical entropy that is finite after renormalization. Unfortunately, the straightforward generalization of this result on higher dimensions meets the still open problem of statistical description of the non-minimally coupled conformal matter.

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